Logic I: Fast Lecture 07

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Readings refer to sections of the course textbook, *Language, Proof and Logic*.

1. Every Time I Go to the Dentist Someone Dies

Reading: §11.2

∀t (

(Time(t) ∧ ToDentist(a,t))

→

∃x (Person(x) ∧ TimeOfDeath(x,t))

)

2. Truth-functional completeness

Reading: §7.4

‘A set of truth-functors is said to be *expressively adequate* (or sometimes *functionally complete*) iff, for every truth-function whatever, there is a formula containing only those truth-functors which express that truth-function, i.e. which has as its truth-table the truth-table specifying that function.’ (Bostock, *Intermediate Logic* p. 45)

Illustration of the proof that {¬, ∧, ∨} is truth-functionally complete:

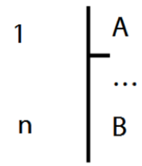
P	Q	P→Q	
T	T	T	[P∧Q] ∨
T	F	F	
F	T	T	[¬P∧Q] ∨
F	F	T	[¬P∧ ¬Q]

[P∧Q] ∨ [¬P∧Q] ∨ [¬P∧ ¬Q]

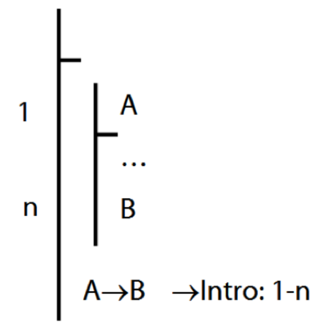
Exercise assuming {¬,∨,∧} is truth-functionally complete, show that {¬,∨} is.

3. Proofs about Proofs

If A ⊢ B then ⊢ A⟶B
Proof Given a proof for A ⊢ B …

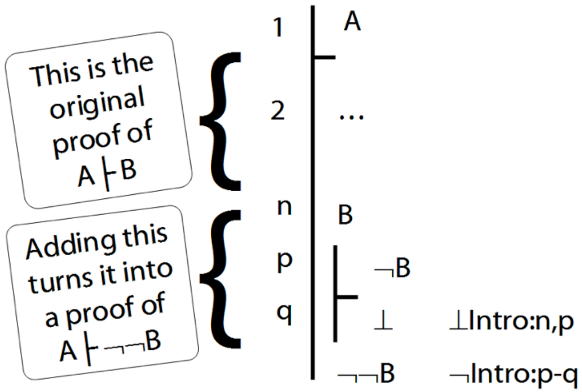


… we can turn it into a proof for ⊢ A⟶B:



If ⊢ A⟶B then A ⊢ B

If A ⊢ B then A ⊢ ¬¬B
Proof:

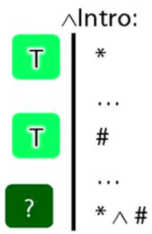


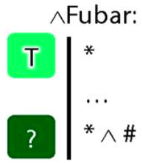
If A ⊢ C then A ⊢ B→C

If A ⊢ B and A ⊢ ¬C then A ⊢ ¬(B→C)

4. The Soundness Property and the Fubar Rules (fast)

Reading: §8.3

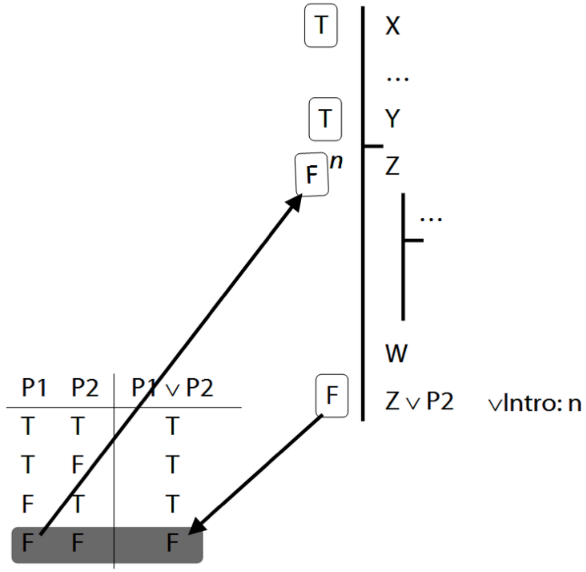




5. Proof of the Soundness Theorem

Reading: §8.3

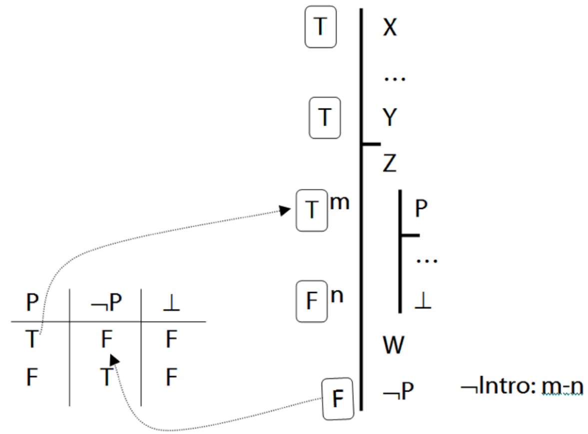
Illustration of soundness proof: ∨Intro



Useful Observation about any argument that ends with ∨Intro. Suppose this argument is not valid, i.e. the premises are true and the conclusion false. Then Z must be false. So Z cannot be a premise. But the argument from the premises to Z (line n) is not a valid argument. So there is a shorter proof which is not valid.

Stipulation: when I say that *a proof is not valid*, I mean that the last step of the proof is not a logical consequence of the premises (including premises of any open subproofs).

Illustration of soundness proof: ¬Intro



How to prove soundness? Outline

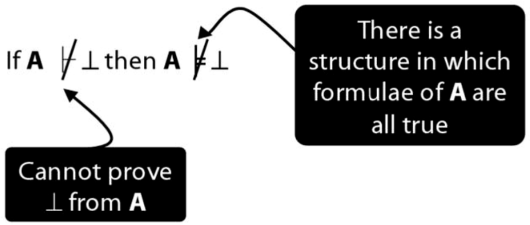
Step 1: show that each rule has this property:

Where the last step in a proof involves that rule, if proof is not valid then there is a shorter proof which is not valid.

Step 2: Suppose (for a contradiction) that some Fitch proofs are not valid. Select one of the shortest invalid proofs. The last step must involve one of the Fitch rules. Whichever rule it involves, we know that there must be a shorter proof which is not valid. This contradicts the fact that the selected proof is a shortest invalid proof.

6. The Essence of the Completeness Theorem

Reading: §8.3



Arrange the sentence letters in a series: P1, P2, P3, …

Define a structure, *h*, as follows.

Take each sentence letter, Pi, in turn.

- If **A** ⊢ Pi then *h*(Pi) = True
- If **A** ⊢ ¬Pi then *h*(Pi) = False
- Otherwise *h*(Pi) = True

Every sentence of **A** is true in the structure *h*

Therefore **A** ⊭⊥

7. Lemma for the Completeness Theorem

Reading: §8.3

If for every sentence letter, P, either A ⊢ P or A ⊢ ¬P, then for every formula, X, either A ⊢ X or A ⊢ ¬X.

Proof

Step a. Suppose (for a contradiction) that there are formulae, X, such that A ⊬ X and A ⊬ ¬X. Take a shortest such formula, call it Y.

Step b. This formula, Y, must have one of the following forms: ¬P, P∨Q, P∧Q, P⟶Q, P⟷Q, ⊥

Step c. We can show that whichever form X has, either A ⊢ Y and A ⊢ ¬Y.

Case 1: X is P⟶Q. Then since P and Q are shorter than X, either:

(i) A ⊢ P and A ⊢ ¬Q

or

(ii) A ⊢ ¬P

or

(iii) A ⊢ Q

If (i), A ⊢ ¬(P⟶Q), that is, A ⊢ ¬X.

If (ii), A ⊢ P⟶Q, that is, A ⊢ ¬X.

If (iii), A ⊢ P⟶Q, that is, A ⊢ ¬X.

(Here we use the last two Proofs about Proofs, see earlier)

Case 2: X is ¬P.

Then since P is shorter, A ⊢ P or A ⊢ ¬P.

If A ⊢ P then A ⊢ ¬¬P so A ⊢ ¬X which would contradict our assumption. This is shown in the proofs about proofs above.

If A ⊢ ¬P then A ⊢ X (because X is ¬P), which would contradict our assumption.

Case 3: …

Step d. The demonstration in Step c contradicts our assumption, so we can conclude that it is false. That is, either A ⊢ X and A ⊢ ¬X for every formula X.

8. Proof of the Completeness Theorem

Reading: §8.3, §17.2

Suppose **A** ⊬⊥. Define **A*** as every formula in **A** plus the following:

For every sentence letter, P,

if **A** ⊢P, add P to **A***

if **A** ⊢¬P, add ¬P to **A***

otherwise add P to **A***

Now **A*** ⊬⊥ and **A*** contains every sentence letter or its negation

Claim: for any formula X, **A*** ⊢X or **A*** ⊢¬X

Define a structure, *h*, so that:

h(P)=T when P is in **A***

h(P)=F when ¬P is in **A***

Claim: *h*(X)=T for every X that is a logical consequence of **A*** (see Prop. 4, p. 475).

Thus **A** ⊭⊥

9. More Records Than the KGB

Reading: §14.1, §14.3

10. The End Is Near

Reading: §14.3