Logic I: Lecture 18

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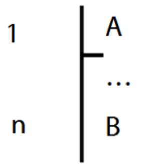
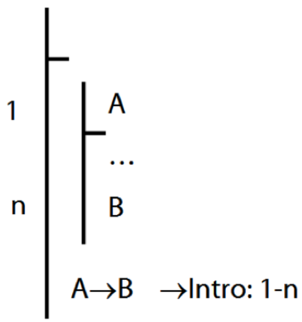
Readings refer to sections of the course textbook, *Language, Proof and Logic*.

1. Proofs about Proofs

If A ⊢ B then ⊢ A⟶B
Proof Given a proof for A ⊢ B …

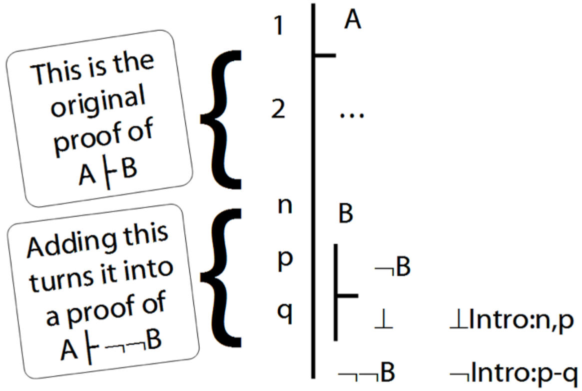


… we can turn it into a proof for ⊢ A⟶B:



If ⊢ A⟶B then A ⊢ B

If A ⊢ B then A ⊢ ¬¬B
Proof:



If A ⊢ C then A ⊢ B⟶C

If A ⊢ B and A ⊢ ¬C then A ⊢ ¬(B⟶C)

2. A ⊨ ⊥

3. If (A ⊬ ⊥ entails A $\nvDash_{TT}$ ⊥) then (A $\vDash_{TT}$ B entails A ⊢ B)

First we show that if A, ¬B ⊢ ⊥ then A ⊢ B .

(To do this, we show how to turn transform a proof of the former kind into a proof of the latter kind.)

Now assume that for all sets **A**, **A** ⊬ ⊥ entails **A** $\nvDash_{TT}$ ⊥

Suppose that **A** $\vDash_{TT}$ B

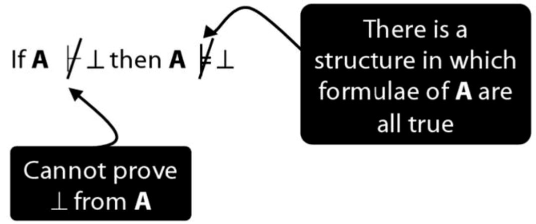
Then **A**, ¬B $\vDash_{TT}$ ⊥

So, by our assumption, **A**, ¬B ⊢ ⊥

Therefore (by the thing we showed first) A ⊢ B

4. The Essence of the Completeness Theorem

Reading: §8.3



Arrange the sentence letters in a series: P1, P2, P3, …

Define a structure, *h*, as follows.

Take each sentence letter, Pi, in turn.

- If **A** ⊢ Pi then *h*(Pi) = True

- If **A** ⊢ ¬Pi then *h*(Pi) = False

- Otherwise *h*(Pi) = True

Every sentence of **A** is true in the structure *h*

Therefore **A** ⊭ ⊥

5. Lemma for the Completeness Theorem

Reading: §8.3

If for every sentence letter, P, either A ⊢ P or A ⊢ ¬P, then for every formula, X, either A ⊢ X or A ⊢ ¬X.

Proof

Step a. Suppose (for a contradiction) that there are formulae, X, such that $A \nvdash X$ and $A \nvdash \neg X$. Take a shortest such formula, call it Y.

Step b. This formula, Y, must have one of the following forms: $\neg P$, $P \vee Q$, $P \wedge Q$, $P \rightarrow Q$, $P \leftrightarrow Q$, $\bot$

Step c. We can show that whichever form X has, either $A \vdash Y$ and $A \vdash \neg Y$.

Case 1: X is $P \rightarrow Q$. Then since P and Q are shorter than X, either:

(i) $A \vdash P$ and $A \vdash \neg Q$

or

(ii) $A \vdash \neg P$

or

(iii) $A \vdash Q$

If (i), $A \vdash \neg(P \rightarrow Q)$, that is, $A \vdash \neg X$.

If (ii), $A \vdash P \rightarrow Q$, that is, $A \vdash \neg X$.

If (iii), $A \vdash P \rightarrow Q$, that is, $A \vdash \neg X$.

(Here we use the last two Proofs about Proofs, see earlier)

Case 2: X is $\neg P$.

Then since P is shorter, $A \vdash P$ or $A \vdash \neg P$.

If $A \vdash P$ then $A \vdash \neg\neg P$ so $A \vdash \neg X$ which would contradict our assumption. This is shown in the proofs about proofs above.

If $A \vdash \neg P$ then $A \vdash X$ (because X is $\neg P$), which would contradict our assumption.

Case 3: …

Step d. The demonstration in Step c contradicts our assumption, so we can conclude that it is false. That is, either $A \vdash X$ and $A \vdash \neg X$ for every formula X.

6. Proof of the Completeness Theorem

Reading: §8.3, §17.1, §17.2

Suppose **A** $\nvdash \bot$. Define **A*** as every formula in **A** plus the following:

For every sentence letter, P,

if **A** $\vdash$ P, add P to **A***

if **A** $\vdash \neg$P, add $\neg$P to **A***

otherwise add P to **A***

Now **A*** $\nvdash \bot$ and **A*** contains every sentence letter or its negation

Claim: for any formula X, **A*** $\vdash$ X or **A*** $\vdash \neg$X

Define a structure, *h*, so that:

h(P)=T when P is in **A***

h(P)=F when $\neg$P is in **A***

Claim: *h*(X)=T for every X that is a logical consequence of **A*** (see Prop. 4, p. 475).

Thus **A** $\nvdash \bot$

7. Proof of Proposition 4 for the Completeness Theorem

Reading: §15.1, §15.6

8. Exercises

These exercises will be discussed in seminars the week after this lecture. The numbers below refer to the numbered exercises in the course textbook, e.g. '1.1' refers to exercise 1.1. on page 39 of the second edition of *Language, Proof and Logic*. Exercises marked '*' are optional.

15.33–15.40 (second edition)